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1995 J. Phys. A: Math. Gen. 28 4733

(http://iopscience.iop.org/0305-4470/28/17/009)

J. Phys. A: Math. Gen. 28 (1995) 4733–4741. Printed in the UK

On the duality between periodic orbit statistics and quantum level statistics

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Received 11 April 1995

Abstract. We discuss consequences of a recent observation that the sequence of periodic orbits in a chaotic billiard behaves like a Poissonian stochastic process on small scales. This enables the semiclassical form factor $K_{sc}(\tau)$ to agree with predictions of random matrix theories for other than infinitesimal τ in the semiclassical limit.

1. Introduction

The spectral form factor $K(\tau)$, defined as the Fourier transform of the spectral autocorrelation function, plays a central role in semiclassical analysis of chaotic systems. All spectral statistics bilinear in the density, such as spectral rigidity (Δ_3) and number variance (Σ_2) may be expressed in terms of the form factor [1, 2]. The form factor is a convenient tool for semiclassical analysis while it can be expressed as a double sum over periodic orbits [1]. The approximation of the form factor thus obtained $K_{sc}(\tau)$ is called the semiclassical form factor. This paper contains some complementary results to Berry's classical paper [1], where he shows that, in the limit of the small τ, the form factor is $K_{sc}(\tau) = \tau$ for system with no time-reversal symmetry and $K_{sc}(\tau) = 2\tau$ for system with time-reversal symmetry. This agrees with predictions from random matrix theories [3] for the Gaussian unitary ensemble (GUE) and Gaussian orthogonal ensemble (GOE) respectively. This result is essentially the only semiclassical support for universality in level statistics for chaotic systems obtained so far. However, as we will see, the result is only obvious for smaller and smaller τ when the semiclassical limit is approached, and we need a mechanism to ensure approximate validity up to fairly large τ (of the order of unity) for universality to be achieved.

In this paper we are going to describe the sequence of periodic orbit in a statistical language and use tools from the thermodynamic formalism of chaotic systems [4]. These tools will be worked out in section 2. Recent numerical results [14] suggests that the sequence of periodic orbits may be described as a Poissonian process, at least on small scales. In section 3 we show that, under this assumption (called assumption B), Berry's result $K = (2)\tau$ may be recovered without assuming conditional convergence of the Gutzwiller formula [5]. And even more important, this result holds for other than infinitesimal τ in the semiclassical limit.

However, we know that assumption B cannot be exactly true. Sooner or later assumption B has to break down and the corresponding deviation from $K = (2)\tau$ may start. In order to produce or real or almost real spectra there simply has to be correlations [23, 24]. But the purpose of the present paper is to explore the consequences of the no correlation assumption and argue that the non-existence, or weakness of correlations between neighbouring periodic orbits is an important mechanism behind the appearance of the form factor.

In section 4 we discuss the various assumptions behind our result and how the results depend on them. In particular we review the numerical evidence for assumption B and in section 5 we discuss related work in view of our achievements.

2. Preliminaries

The key step is the analytical continuation of the Gutzwiller formula for the level density down to the real energy axis. We are going to perform this for the Fourier transform of the level density under some simplifying assumptions.

2.1. The Fourier transform of the level density

Our starting point is Gutzwiller–Voros zeta function [5, 6], which for systems with two degrees of freedom, reads

$$Z(E) = \prod_p \prod_{j=0}^{\infty} \left(1 - \frac{\mathrm{e}^{\mathrm{i}S_p(E)}}{|\Lambda_p|^{1/2}\Lambda_p^j}\right) \tag{1}$$

where Λ_p is the expanding eigenvalue of the Jacobian along the primitive periodic orbit p. To make things as transparent as possible we restrict ourselves to billiards, so that the action integral is simply $S_p(E) = l_p \cdot \kappa(E)$ where the 'momentum' is $\kappa = \sqrt{2E}$ and l_p the length of orbit p. The level density to be discussed is measured in κ-space and not, as is usual, in E-space. We use units such that $m = \hbar = 1$. We have neglected the Maslov indices: this is discussed in subsection 2.2 and section 4.

It is expected that the zeros of the zeta function approximate the quantum eigenvalues. How the semiclassical approximation and the stationary phase approximation behind Z_{GV} affect the spectrum is not entirely clear. However, this is not our main concern here. What is important in this paper is that the zeta function may contain additional zeros reflecting its convergence properties.

In this paper we will use the following simplifying assumption.

Assumption A. *The zeta function $Z(\kappa)$ is entire and has zeros on the exact quantum* positions $\pm\kappa_i$, $i \neq 0$ and one extra zero $\kappa_0 = \mathrm{i}h_{1/2}$ on the positive imaginary axis lying on *the border of convergence.*

The motivation and discussion of this assumption is postponed until section 4. (The reason behind the choice of subscript of $h_{1/2}$ will become obvious in subsection 2.2.)

The level density is usually split up into the mean density and an oscillating part $d(\kappa) = d_0(\kappa) + d_{\mathrm{osc}}(\kappa)$. The leading part of d_0 is the Weyl term $d_0 \sim A\kappa/2\pi$ where A is the billiard area. The Gutzwiller–Voros zeta function above is derived from the Gutzwiller formula for d_{osc}. For later purposes we need the Fourier transform of the oscillating part of the level density

$$\tilde{D}_{\mathrm{osc}}(l) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty+\mathrm{i}C}^{\infty+\mathrm{i}C} \mathrm{e}^{-\mathrm{i}\kappa l} \frac{\mathrm{d}}{\mathrm{d}\kappa} \log Z(\kappa)\, \mathrm{d}\kappa\,. \tag{2}$$

We will evaluate it in two ways. First, by inserting the product representation of the zeta function (1). It is thus essential that the constant C in (2) is sufficiently large so that the contour runs in the region where the product (1) converges (and thus well above all zeros of Z). One can then exchange summation and integration and establish the result

$$\tilde{D}_{\mathrm{osc}}(l) = \sum_p l_p \sum_{n=1}^{\infty} \frac{\delta(l - nl_p)}{|M_p^n - I|^{1/2}} \,. \tag{3}$$

Exponential divergence of this sum $\tilde{D}_{\mathrm{osc}}(l) \to \exp(h_{1/2}l)$, a feature to be discussed in subsection 2.2, is directly related to the presence of the zero $\kappa_0 = \mathrm{i}h_{1/2}$ as can be seen from (2)

Secondly we compute $\tilde{D}_{\mathrm{osc}}$ by means of residue calculus

$$\tilde{D}_{\mathrm{osc}}(l) = \mathrm{e}^{h_{1/2}l} + \sum_{j=-\infty, j\neq 0}^{\infty} \mathrm{e}^{-\mathrm{i}\kappa_j l} \,. \tag{4}$$

The reader may wonder why the contribution from the mean distribution d_0 has disappeared in (4). The reason is that we have neglected the contribution from the large semi-circle which would have yielded delta functions, and even more nasty things, associated with the Fourier transform of d_0. The tilde-sign in $\tilde{D}_{\mathrm{osc}}$ indicates that the result carries contribution from the extra zero κ_0. Without this zero we get the Fourier transform of the true level density $D_{\mathrm{osc}} = \sum_{j=-\infty, j\neq 0}^{\infty} \mathrm{e}^{-\mathrm{i}\kappa_j l}$. We can then establish the identity

$$D_{\mathrm{osc}}(l) = \sum_p l_p \sum_{n=1}^{\infty} \frac{\delta(l - nl_p)}{|M_p^n - I|^{1/2}} - \mathrm{e}^{h_{\mathrm{sc}}l} \,. \tag{5}$$

We have, all the way, assumed that $l > 0$.

From now on we will tacitly replace every occurrence of $|M_p^n - I|$ with $|\Lambda_p|$. The errors thus induced are completely negligible in the limits we are going to explore. We must now discuss some properties of the periodic orbit sums.

2.2. *Some properties of the set of periodic orbits*

Essentially all dynamical information is encoded in the sequence of the invariants of the primitive periodic orbits $\{l_p, \Lambda_p\}$ (the Maslov indices provide some topological information, however). In the asymptotic limit $l \to \infty$ one can establish the following family of sum-rules for chaotic systems:

$$\sum_p l_p \sum_{n=1}^{\infty} \frac{\delta(l - nl_p)}{|\Lambda_p|^{\beta}} \to \mathrm{e}^{h_\beta l} \tag{6}$$

valid at least for $\beta \leqslant 1$ [9]. (From now on we will reside in the asymptotic limit of large l and write equality signs instead of arrows.) The result applies after appropriate smearing of the delta functions. The entropy-like quantity h_β decreases with increasing β. The special case $\beta = 1$ has already been discussed in [7]; for bound systems one has $h_1 = 0$. h_0 is the topological entropy [8]. General β are discussed in [8, 9, 10], for example.

It is very useful to describe the set of periodic orbits in a statistical language. For large l the repetitions of shorter orbits are overwhelmed by the number of primitive orbits so that we may neglect the sum over n above. Let us call the density of of prime orbits $\phi(l) = \sum_p \delta(l - l_p)$. From (6) we see that the mean value of this density is

$$\langle \phi(l) \rangle = \mathrm{e}^{h_0 l}/l \,. \tag{7}$$

For general β we express the sum rules in terms of the averages $\langle |\Lambda_p|^{-\beta} \rangle$:

$$l \langle \phi \rangle \langle |\Lambda|^{-\beta} \rangle = \mathrm{e}^{h_0 l} \langle |\Lambda|^{-\beta} \rangle = \mathrm{e}^{h_\beta l} \,. \tag{8}$$

For instance, we get the result (which will be of use later)

$$\langle|\Lambda|^{-1}\rangle = e^{-h_0 l} \,. \tag{9}$$

So much for the large-scale structure of the sequence $\{l_p, \Lambda_p\}$, i.e. large smearing widths in (6). What about the small-scale structure?

Let us order this sequence according to increasing l_p and consider the ordered sequence $\{l_i, \Lambda_i\}$ where the integer i denote the position in the sequence. Then rescale the length variable according to

$$\ell_i = \int_0^{l_i} \langle\phi(l')\rangle \, dl' \tag{10}$$

so that the mean spacing $\langle \ell_i - \ell_{i-1}\rangle$ is unity. We will now make our main assumption.

Assumption B. (i) The sequence ℓ_i is given by a Poissonian process with unit intensity. *(ii) The corresponding stabilities Λ_i may be considered as mutually independent stochastic variables.*

We discuss the evidence for this assumption in section 4.

We can now reformulate D_{osc} in a purely statistical language

$$D_{osc}(l) = l\left(\phi(l)\frac{1}{\sqrt{|\Lambda(l)|}} - \langle\phi\rangle\left\langle\frac{1}{\sqrt{|\Lambda|}}\right\rangle\right) . \tag{11}$$

The introduction of phase indices (Maslov indices and symmetry indices [12]) will generally move down the leading zero $h_{1/2}$ [10] and there is a possibility that it might even cross the real κ-axis, making the Gutzwiller sum conditionally convergent [11]. It is non-trivial to deduce if this really takes place for a given system. Many estimations in the literature of the position of the *entropy barrier*, with or without Maslovs, assume uniform hyperbolicity of the system and are thus invalid for generic systems.

3. The form factor

The spectral form factor is defined as the Fourier transform of the spectral autocorrelation function

$$K = \frac{1}{d_0}\int_{-\infty}^{\infty} d\epsilon \; e^{-i\epsilon l} d_{osc}(\kappa + \epsilon/2) d_{osc}(\kappa - \epsilon/2) \,. \tag{12}$$

It is usually regarded as a function of the dimensionless length variable $\tau = l/(2\pi d_0) = l/(A\kappa)$ with κ as a parameter. The suggested universal behaviour of $K(\tau)$ should arise in the semiclassical limit $\kappa \to \infty$. To be meaningful the form factor needs some averaging which we will apply first at the end. The form factor can be expressed in terms of the Fourier transform of $d_{osc}(\kappa)$ according to

$$K = \frac{1}{2\pi d_0}\int_{-\infty}^{\infty} dl' \int_{-\infty}^{\infty} dl'' \; \delta\left(l - \frac{l' + l''}{2}\right)\cos(\kappa(l' - l''))D_{osc}(l')D_{osc}(l'') \,. \tag{13}$$

The derivation is straightforward: one has to use the fact that $d_{osc}(\kappa)$, and thus $D_{osc}(l)$, are real and even.

Assuming conditional convergence

Let us now follow Berry's arguments a little further [1]. If the Gutzwiller formula is conditionally convergent we can insert $\tilde{D}_{\rm osc}$ directly instead of $D_{\rm osc}$ into (13):

$$K_{\rm sc} = \frac{l^2}{2\pi d_0} \int\int {\rm d}l'\,{\rm d}l''\, \cos(\kappa(l'-l''))\delta\left(l - \frac{l'+l''}{2}\right)\left(\sum_i \frac{\delta(l'-l_i)}{\sqrt{|\Lambda_i|}}\right) \times \left(\sum_j \frac{\delta(l''-l_j)}{\sqrt{|\Lambda_j|}}\right). \tag{14}$$

By keeping κ constant and letting $l \to 0$ be small, the cosine will wash away the non-diagonal terms (assuming no systematic degeneracies between the l_i due to, e.g., time-reversal symmetry) and we get

$$K_{\rm sc} = \frac{l^2}{2\pi d_0}\left(\sum_i \frac{\delta(l-l_i)}{|\Lambda_i|}\right). \tag{15}$$

Using the sum rules in subsection 2.2 we get

$$\bar{K} = \frac{l^2}{2\pi d_0}\frac{1}{l}\exp(-h_0 l) = \frac{l}{2\pi d_0} = \tau\,. \tag{16}$$

This result gave rise to some enthusiasm since it agrees with predictions of random matrix theories. But one may ask two questions.

First, is equation (16) true even if the Gutzillwer sum is not conditionally convergent?

Second, we note that the number of periodic orbits contained in one period of the cosine in (14) is $\sim \langle\phi\rangle/\kappa \sim \exp(h_0 A\kappa\tau)/(A\kappa^2\tau)$. To make this estimate we have assumed that the smearing width is $\Delta l \approx \langle\phi\rangle^{-1}$, the smallest conceivable choice. In the semiclassical limit ($\kappa \to \infty$) the argument leading to (16) is brutally violated for other than infinitesimal τ. If the predictions of random matrix theories are correct, one expects equation (16) to hold up to $\tau \sim 1$. Now to the second question. Is the result $K = \tau$ correct for finite τ, and if the answer is yes, why?

The generic case

The key lies in the stochastic nature of the periodic orbits as formulated in assumption **B**. First we insert the general formula for $D_{\rm osc}(l)$:

$$K_{\rm sc} = \frac{l^2}{2\pi d_0}\int\int {\rm d}l'\,{\rm d}l''\,\cos(\kappa(l'-l''))\delta(l - \frac{l'+l''}{2}) \times \left(\sum_i \frac{\delta(l'-l_i)}{\sqrt{|\Lambda_i|}} - \langle\phi\rangle\left\langle\frac{1}{\sqrt{|\Lambda|}}\right\rangle\right)\left(\sum_j \frac{\delta(l''-l_j)}{\sqrt{|\Lambda_j|}} - \langle\phi\rangle\left\langle\frac{1}{\sqrt{|\Lambda|}}\right\rangle\right). \tag{17}$$

We see clearly that we are dealing with correlation in the sequence $\{l_i, \Lambda_i\}$ and, according to assumption B, there are no correlations at all. Therefore only the diagonal terms will contribute. In order to avoid delta functions in the resulting form factor we smear it:

$$\bar{K}_{\rm sc} = \frac{1}{\Delta}\int_l^{l+\Delta} K(l')\,{\rm d}l' \tag{18}$$

which may now be expressed as

$$\bar{K}_{\rm sc} = \frac{l^2}{2\pi d_0 \Delta}\int_l^{l+\Delta} {\rm d}l' \left\{\left(\sum_i \frac{\delta(l'-l_i)}{\sqrt{|\Lambda_i|}} - \langle\phi\rangle\left\langle\frac{1}{\sqrt{|\Lambda|}}\right\rangle\right)\right\}^2. \tag{19}$$

The integral in this expression is just the variance of the sum of $1/\sqrt{|\Lambda|}$ in a window of a Poissonian process:

$$\bar{K} = \frac{l^2}{2\pi d_0 \Delta} V_\Delta \left(\sum \frac{1}{\sqrt{|\Lambda|}} \right) . \tag{20}$$

The calculation of this variance is an elementary exercise in probability theory (see the appendix) and the result is $V_\Delta = \lambda\Delta\langle 1/\sqrt{|\Lambda|}^2 \rangle$. The intensity λ equals the mean density of prime orbits $\lambda = \langle \phi(l) \rangle = \exp(h_0 l)/l$ and, according to the sum rules in subsection 2.2, we have $\langle 1/\sqrt{|\Lambda|}^2 \rangle = \langle 1/\Lambda \rangle = \exp(-h_0 l)$. We thus get our final result:

$$\bar{K}_{\mathrm{sc}} = \frac{l^2}{2\pi d_0 \Delta} \frac{\exp(h_0 l)\Delta}{l} \exp(-h_0 l) = \frac{l}{2\pi d_0} = \tau \tag{21}$$

which is the same as for conditionally convergent systems but the result now holds for other than infinitesimal τ in the semiclassical limit.

It is straightforward to generalize to the systematic degeneracies of the periodic orbit exhibited by time-reversible systems, and we will not discuss this here.

4. Motivations for our basic assumptions

Our result relies on a series of assumptions and approximations. Some of them may be removed or modified without altering the result. In this section we give our motivations for our assumptions and discuss the extent to which the results depend on them.

First assumption A. The presence of a leading zero $\kappa_0 = \mathrm{i}h_{1/2}$ such that $h_{1/2} > 0$ has already been discussed. In the general case it is reasonable to assume the presence of several zeros not associated with any quantum state. Their contribution is naturally included into $\langle \phi \rangle \langle |\Lambda|^{-1/2}(l) \rangle$, giving rise to oscillatory and exponentially decreasing corrections.

The assumption that the semiclassical zeros equal the quantum eigenvalues is not crucial for our results. It was mainly introduced for computational and notational convenience. However, the expected failure of the semiclassical eigenvalues to be real will have consequences for the large-τ behaviour of the semiclassical form factor [19]: see section 5!

There is one example for which assumption A is fulfilled exactly, and that is compact billiards on surfaces of constant negative curvature. These systems are special, having zeros on the exact quantum positions. But there is also a zero on the imaginary κ axis, right on the border of convergence [13]. Indeed there is a zero on the border of convergence of each j-factor in the zeta function (1).

Now to assumption B. In [14] the authors pursued the original idea of considering the spectrum of lengths l_j of the prime cycles and performed level statistics *à la* quantum chaos. Their system was a touching three-disk billiard. The first step is to unfold the spectrum, cf subsection 2.2, yielding the sequence ℓ_j. Then they studied the level spacing distribution, which was found to be an exponential to very high degree of accuracy, and spectral rigidity, which was found to agree with $\Delta_3(L) = L/15$. This is consistent with the sequence ℓ_j being given by a Poissonian process and motivates assumption B. The three-disk billiard, as well as any generic Euclidean chaotic billiard, has almost certainly an infinite symbolic dynamics. It would be nice to know if the greater regularity between the cycles for a finite symbolic dynamics still exhibits this kind of randomness.

We do not attempt to explain this random feature; rather we only offer the following hand-waving argument. Neighbours in the sequence $\{l_j\}$ need not be close in phase space and there is therefore no reason for correlations. In a small proximity to a given length l there may be many periodic orbits. The sequence of periodic orbit could thus, perhaps, be viewed as the random superposition of many sequences, each sequence corresponding to a topologically distinct family of periodic orbits. This would give rise to the Poissonian nature.

Assumption B(ii), concerning the the stabilities, is in the same spirit as B(i). If the lengths are uncorrelated so should the eigenvalues.

We again stress that we do not expect this assumption to hold thoughout the sequence of periodic orbtis, for reasons mentioned in the introduction. We expect long-range correlations and perhaps weak short-range correlations. The latter may explain why the GOE form factor only *approximately* equals 2τ for $\tau < 1$ whereas the GUE form factor exactly equals τ. It could be due to weak correlations, unseen in [14], which would effectively be washed away by, e.g., a magnetic field breaking the symmetry.

The restriction to billiards is mostly for convenience. We find it highly unlikely that a smooth potential would exhibit complete chaos [25]. Our calculations would easily be modified for a chaotic smooth potential, if it exists, which is homogeneous; the 'momentum' variable κ would be some other power of the energy E. If one chooses to consider non-homogeneous potentials one must perform the Fourier transform with respect to $\hbar$ instead of κ We will not speculate about this case, since it would take us too far from the case where assumption B has been verified.

5. Discussion

It is interesting to note that the semiclassical form factor $K_{sc}(\tau)$ in the region $0 < \tau < 1$ depends on *both* the very-small-scale structure *and* the very coarse structure on the sequence of periodic orbits. In this paper we have focussed on the deep asymptotic limit $\kappa \to \infty$. For finite energies one has to consider pre-asymptotic behaviour and power-law corrections of the periodic orbit sum rules involved. This is discussed in [18, 10]. In these papers we did not correct for the divergence of the trace formula, but this procedure is readily justified from the present results. This pre-asymptotic behaviour considerably extends the non-universal regime in spectral statistics derivable from the form factor compared with the regime discussed by Berry [1]. Such non-universal regimes have been observed in several numerical experiments [16, 17, 15, 10]. In [10] we also discuss the role of marginally stable orbits, which plays a major role for moderate energies.

The semiclassical status of the large τ limit is much more obscure. It is clear that the quantum form factor approaches unity in this limit $K \to 1$ provided that there are no systematic degeneracies in the spectrum. In [19] Keating demonstrates that, since we cannot expect the trace formula to produce poles exactly on the real axis, the semiclassical form factor should diverge exponentially. If the imaginary part of the poles is much less than the mean spacing, as is indicated by [20], this exponential take-off should not occur until fairly large τ, and one can still hope for saturation of the semiclassical form-factor.

Some evidence of saturation is presented in [21] for the hyperbola billiard and other systems.

The exponential collapse of $K_{sc}(\tau)$ reported in [22] appears to be due to neglect of the extra zero(s), and illustrates the hazard of inserting diverging series into the form-factor. This is particularly dangerous, since the expression thus obtained need not be divergent.

Whatever happens to the form factor, we expect deviation from $K_{sc}(\tau) = (2)\tau$. If the expected saturation indeed takes place it is, of course, highly desirably to understand its classical origin, its manifestation by the periodic orbits, and the connection with the fact that the system is bound.

The reader may think that our assumption B contradicts the concept of *action repulsion* discussed by Argaman *et al* [21, 19]. However, if the periodic orbit correlations proposed in [21] really occur, this repulsion is not an effect acting between neighbours in the sequence $\{l_i\}$ of cycles but over vast distances; the name action repulsion might thus be misleading. The authors of [21] stress that the proposed correlation between periodic orbits is a weak effect superposed on a Poissonian background and there is no contradiction between our results.

Much of the present work on the semiclassical trace formula is concerned with taming the diverging trace formula and the computation of corrections, in order to obtain accurate results for the bottom part of the spectrum. Many of these corrections disappear in the semiclassical limit. In this paper we took the opposite point of view. We tried to relate asymptotic (= semiclassical) properties of the spectrum to the asymptotic behaviour of the periodic orbits. As both are conveniently expressed in a statistical language this approach aims at unveiling the duality between periodic orbit statistics and level statistics.

Appendix

Åke Svensson owns a small shop in the old town of Stockholm. Customers arrive at the shop according to a Poissonian process with intensity λ. The amount of money paid by one customer is considered to be a stochastic variable x with probability distribution $f(x)$. The x belonging to different customers are mutually independent.

During a time T the amount of cash received by Åke is X_T. What is the variance of X_T?

The distribution of arrivals during time T in a Poissonian process is

$$p_n = e^{-\lambda T}\frac{(\lambda T)^n}{n!} \tag{A1}$$

so the distribution of the variable X is

$$F(X) = \sum_{n=0}^{\infty} p_n f^{*n}(X) \tag{A2}$$

where f^{*n} is the n-fold convolution of f. The mean and variance in such a convolution are additive so we have

$$\langle x_n\rangle = n\langle x\rangle \tag{A3}$$

$$\langle x_n^2\rangle = n\langle x^2\rangle + n(n-1)\langle x\rangle^2 . \tag{A4}$$

A short calculation now yields the mean and variance of X_T to be

$$\langle X_T\rangle = \lambda T\cdot\langle x\rangle \tag{A5}$$

$$V(X_T) \equiv \langle X_T^2\rangle - \langle X_T\rangle^2 = \lambda T\cdot\langle x^2\rangle . \tag{A6}$$

Acknowledgments

I would like to thank Jon Keating for valuable comments. This work was supported by the Swedish Natural Science Research Council (NFR) under contract No F-FU 06420-303.

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